Loss unit interpolation in the collateralized debt obligation pricing model

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Abstract

In this paper I present a small change to the updating formula in Andersen, Sidenius and Basu's recursion method for calculating loss distributions. The change allows accurate representation of the losses of very heterogeneous portfolios with a small number of loss states, improving performance. Also, the inverse formula is given, for fast calculation of spread sensitivities.

1 Standard recursion

In their paper, Andersen, Sidenius and Basu[1] present a method for building up the loss distribution of independently defaulting obligors by starting with the loss distribution of an empty pool ($P_0 = 1$ and $P_i = 0$ for all $i > 0$). The loss distribution is discretized, so that only integer multiples of a "loss unit" are possible. Obligors are added to the pool sequentially. The loss distribution of the pool with the obligor added, P' is modelled for each addition by applying the updating formula to the loss distribution without the obligor added, P:

$$P'_i = \begin{cases} (1-p)P_i & \text{if } i < l \\ (1-p)P_i + pP_{i-l} & \text{if } i \geq l \end{cases}$$

where p is the probability of the obligor defaulting, and l is the number of units of loss incurred on default.

The method requires the choice of loss unit, with the losses of each obligor represented as an integer multiple of the loss unit. This works well for many CDOs, including the important case of the CDX and iTraxx indices, where the notionals are equal and it is standard to assume equal recovery rates. However, for very heterogeneous portfolios, the choice can be between a very small loss unit which slows computations, or large errors representing losses on some obligors and a corresponding effect on final numerical results.

One method for tackling this problem is given by Hull and White[2]: the mean loss for each "loss bucket" conditional on being in that bucket is calculated along with the probability of being in that bucket.

2 Loss unit interpolation

As an alternative, consider an obligor that has two possible outcomes: no default, or default with x units of loss incurred, with x not necessarily integer. This obligor can be represented approximately as an obligor that has three possible outcomes: no default, default with $\lfloor x \rfloor$ units of loss or $\lceil x \rceil$ loss units of loss. Here $\lfloor x \rfloor$ and $\lceil x \rceil$ are the nearest integers below and above x, respectively.

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We choose the probabilities of these two events so that the expected amount of loss is preserved. In particular, given a default, $\lfloor x \rfloor$ units of loss occur with probability $1 - [x]$ and $\lceil x \rceil$ units of loss occur with probability $[x]$, where $[x] = x - \lfloor x \rfloor$ is the fractional part of x.

For example, an obligor with a 0.05 probability of defaulting, with 2.2 units of loss incurred on default would become an obligor with probability 0.95 of incurring 0 loss units, 0.04 of incurring 2 loss units and 0.01 of incurring 3 loss units.

The updating formula becomes:

$$P'_i = \begin{cases} (1-p)P_i & \text{if } i < \lfloor l \rfloor \\ (1-p)P_i + pP_{i-\lfloor l \rfloor}(1-[l]) & \text{if } i = \lfloor l \rfloor \\ (1-p)P_i + pP_{i-\lfloor l \rfloor}(1-[l]) + pP_{i-\lfloor l \rfloor - 1}[l] & \text{if } i > \lfloor l \rfloor \end{cases} \tag{1}$$

Note that this reduces to the standard recursion in the case that $[l] = 0$. In this case the standard recursion can be used for slightly higher performance.

This approximation is not perfect. While the expected loss of a portfolio is preserved, the expected loss of any particular tranche will not be. To give an extreme example, representing 100 obligors with losses of 0.01 loss unit on default, this method would give a positive probability of 100 loss units occurring, whereas we know a priori that the largest possible loss is 1 unit.

In practice the method seems reasonably robust, and will perform at least as well as the basic recursion method given the same loss unit. Further, we can take some assurance from the fact that in the example above, the positive probabilities assigned to more than 1 unit of loss would be small. However, we should be aware that the choice of loss unit, although less critical, is still important in determining numerical accuracy of final results.

3 Inverse formula

For the computation of sensitivities to spread movements, it is useful to have an inverse to (1) which allows an obligor to be removed from a pool. In this way, once the loss distributions necessary for a PV calculation have been performed, delta calculation for an obligor (and for all obligors) can be performed by removing the obligor from the loss distributions, and then adding it back in with probabilities calculated using perturbed spreads. This is a significant computational saving over recalculating the loss distribution from scratch for each delta.

For calculation, it is useful to take $P_i = 0$ for $i < 0$ to reduce (1) to

$$P'_i = (1-p)P_i + pP_{i-\lfloor l \rfloor}(1-[l]) + pP_{i-\lfloor l \rfloor - 1}. \tag{2}$$

Then, in the general case, rearranging gives the inverse recursion

$$P_i = \frac{P'_i - pP_{i-\lfloor l \rfloor}(1-[l]) - pP_{i-\lfloor l \rfloor - 1}}{1-p}.$$

Note that the calculation of P_i depends on the previous calculations of P_j for $j < i$. If $1-p$ is close to 0 this can cause an instability. In this case, we assume $p = 1$ exactly in (2), giving the alternative inverse recursion:

$$P_i = \frac{P'_{i+\lfloor l \rfloor} - P_{i-1}[l]}{1-[l]}$$

The general case also breaks down when $\lfloor l \rfloor = 0$. In this case, substituting $\lfloor l \rfloor = 0$ in (2) and rearranging gives:

$$P_i = \frac{P'_i - pP_{i-1}[l]}{1-p[l]}$$

Taking all of these cases together gives:

$$P_i = \begin{cases} \frac{P'_i}{1-p[l]} & \text{if } \lfloor l \rfloor = 0, i = 0 \\ \frac{P'_i - pP_{i-1}[l]}{1-p[l]} & \text{if } \lfloor l \rfloor = 0, i > 0 \\ \frac{P'_{i+\lfloor l \rfloor}}{1-[l]} & \text{if } \lfloor l \rfloor > 0, p \approx 1, i = 0 \\ \frac{P'_{i+\lfloor l \rfloor} - P_{i-1}[l]}{1-[l]} & \text{if } \lfloor l \rfloor > 0, p \approx 1, i > 0 \\ \frac{P'_i}{1-p} & \text{if } \lfloor l \rfloor > 0, p \not\approx 1, i < \lfloor l \rfloor \\ \frac{P'_i - pP_{i-\lfloor l \rfloor}(1-[l])}{1-p} & \text{if } \lfloor l \rfloor > 0, p \not\approx 1, i = \lfloor l \rfloor \\ \frac{P'_i - pP_{i-\lfloor l \rfloor}(1-[l]) - pP_{i-\lfloor l \rfloor-1}}{1-p} & \text{if } \lfloor l \rfloor > 0, p \not\approx 1, i > \lfloor l \rfloor \end{cases}$$

4 Numerical results

We tested the model on a realistic 5-year CDO deal with 150 obligors with diverse notionals and recovery rates, a pool notional of $1bn and a 12%-13% tranche using 50 abscissas for the numerical integration. The perfect loss unit (i.e. the largest common divisor of possible losses on default for each obligor) for this deal is $5,000. The average loss on default is $4.16m and the standard deviation is $2.85m. The largest loss on default is $12m, and the smallest is $60k.

Loss unit ($k)	30	60	120	300	600	1,200
Obligors represented inaccurately	12	15	45	17	46	102
Total error representing loss ($k)	125	315	2,345	1,695	11,775	47,395
Fair spreads (bps)						
New recursion	55.151	55.151	55.152	55.150	55.164	55.233
Standard recursion	55.080	55.015	54.633	54.245	52.287	41.760
Time to compute (seconds)						
New recursion	144.9	74.2	40.9	18.9	11.8	8.1
Standard recursion	142.9	72.2	38.0	18.0	11.0	7.6

Figure 1: Fair spreads calculated for selected loss units with the standard and new recursion methods.

The fair spreads calculated with the standard and new recursion methods are shown in figure 1. It can be seen that the standard recursion is very sensitive to the choice of loss unit. Note that the fair spreads calculated by the standard recursion approach those of the new recursion as the loss unit decreases. However, the new method is able to calculate fair spread to within a basis point even when the majority of the obligors are represented with non-integer loss units.

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References

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